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DETERMINANTS OF LAPLACIANS ON THE SPACE OF CONICAL METRICS ON THE SPHERE

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ABSTRACT. On a compact surface with smooth boundary, the determinant of the Laplacian associated to a smooth metric on the surface (with Dirichlet boundary conditions if the boundary is nonempty) is a well-defined isospectral invariant. As a function on the moduli space of such surfaces, it is a smooth function whose boundary behavior in certain cases is well understood; see [OPS and K]. In this paper, we restrict ourselves to a certain class of singular metrics on closed surfaces called conical metrics. We show that the determinant of the associated Laplacian is still well defined and that it is a real analytic function on a suitably restricted subset of the space of conical metrics on the sphere.

1. INTRODUCTION

Let Σ denote a compact surface without boundary. Let σ be a smooth metric on Σ and Δ be its Laplace-Beltrami operator acting on functions. Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots$ denote the eigenvalues of Δ. Then for an orthonormal basis of functions $\{u_j\}$ we have $\Delta u_j + \lambda_j u_j = 0$. We recall that the determinant of the Laplacian, $\det' \Delta$, is formally defined as

$$\det' \Delta = \prod_{j=1}^{\infty} \lambda_j ,$$

and that to give meaning to this product we use the standard zeta regularization. We introduce the zeta function

$$Z(s) = \sum_{j=1}^{\infty} \lambda_j^{-s} \tag{1.1}$$

in terms of which

$$\det' \Delta = \exp(-Z'(0)). \tag{1.2}$$

It follows from the asymptotic distribution of the λ_j 's, given by Weyl's law, that $Z(s)$ defines an analytic function for $\Re(s) > 1$. To study the analytic continuation of $Z(s)$, we note that since the gamma function satisfies

$$\Gamma(s)a^{-s} = \int_0^{\infty} e^{-at} t^{s-1}\, dt ,$$

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we have for $\Re(s) > 1+\varepsilon$,

$$\Gamma(s)\sum_{j=1}^{\infty}\lambda_j^{-s} = \sum_{j=1}^{\infty}\Gamma(s)\lambda_j^{-s}.$$

Thus, for $\Re(s)$ large we can write

$$Z(s) = \frac{1}{\Gamma(s)}\int_0^{\infty}\mathrm{TR}\left(e^{\Delta t} - \frac{1}{A}\right)t^s\frac{dt}{t}, \tag{1.3}$$

with A being the area of (Σ, σ).

It is easy to see that

$$\int_1^{\infty}\left(\sum_{j=1}^{\infty}e^{-\lambda_j t}\right)t^{s-1}\,dt$$

can be analytically continued in s, to all of $\mathbf{C}$. Also as is well known, see [MS], the kernel of $e^{\Delta t}$ has the following expansion as $t\to 0$;

$$\sum_{j=0}^{\infty}e^{-\lambda_j t}u_j^2(x) = \frac{1}{4\pi t} + \frac{K(x)}{12\pi} + O(t), \tag{1.4}$$

where $K(x)$ is the Gaussian curvature of Σ at x. Integrating these local invariants over Σ yields as $t\to 0$,

$$\mathrm{TR}(e^{\Delta t}) = \frac{A}{4\pi t} + \frac{\chi(\Sigma)}{6} + O(t), \tag{1.5}$$

where $\chi(\Sigma)$ denotes the Euler characteristic of Σ. It follows that

$$Z(s) = \frac{1}{\Gamma(s)}\left\{\frac{A}{4\pi(s-1)} + \left(\frac{\chi(\Sigma)}{6} - 1\right)\frac{1}{s} + \text{analytic in } s\right\} \tag{1.6}$$

in the region $\Re(s) > -1$. Hence Z has an analytic continuation as a meromorphic function, is regular at $s = 0$ and (1.2) makes sense.

In case Σ has smooth boundary, we subject Δ to Dirichlet boundary conditions and use the analogue of (1.5) to show that (1.2) still makes sense. We note that to define $\det'\Delta$ using (1.2) we needed the discreteness of the spectrum, $\mathrm{spec}(\Delta)$, the asymptotic distribution of the eigenvalues and some known results about the small time behavior of the heat kernel.

From here on Σ will denote a closed surface of genus p. Let $\{P_1, \dots, P_n\}$ be n distinct points on Σ with $n \geq 3$ for $p = 0$ and $n \geq 1$ for $p \geq 1$. A *conical metric* on Σ with vertices $\{P_1, \dots, P_n\}$ and exponents $\{a_1, \dots, a_n\}$ is a flat metric on $\Sigma\backslash\{P_1, \dots, P_n\}$ such that in the neighborhood of each P_i there are isothermal coordinates so that the line element takes the form

$$ds = |w|^{a_i}|dw| \tag{1.7}$$

in $0 < |w| < \delta_i$, for some $\delta_i > 0$, where

$$\text{(a)}\quad a_i > -1, \qquad \text{(b)}\quad \sum_{i=1}^{n}a_i = 2p-2. \tag{1.8}$$

We need condition (1.8(a)) to ensure that the line element (1.7) is integrable on smooth curves through the vertices and condition (1.8(b)) follows the Gauss-Bonnet theorem. We shall see that we can still use (1.2) to define $\det'\Delta$, when Δ is the Laplacian associated to a conical metric on Σ.

Next we say two conical metrics σ_1 on $\Sigma\backslash\{P_1, \dots, P_n\}$ and σ_2 on $\Sigma\backslash\{Q_1, \dots, Q_n\}$ are equivalent if there is a sense-preserving diffeomorphism f onto Σ onto itself mapping vertices to vertices for which $\sigma_2 = f^*\sigma_1$. The space of conical metrics on Σ, $\mathscr{C}_n$, is defined to be

$$\mathscr{C}_n = \{\text{conical metrics}\}/\text{Diff}^+(\Sigma),$$

with the action of the diffeomorphism group as above. Let $\mathscr{C}_n^*$ be the subset of $\mathscr{C}_n$ consisting of those metrics with fixed exponents. We show that, for $\Sigma = S^2$,

Theorem. $\det'\Delta$ *is real analytic on* $\mathscr{C}_n^*$.

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2. Conic Laplacian

Let σ be a conical metric on Σ with vertices $\{P_1, \dots, P_n\}$ and exponents $\{a_1, \dots, a_n\}$. In geodesic polar coordinates, this metric takes the form

$$ds^2 = dr^2 + (1+a_i)^2 r^2\, d\theta^2, \qquad r = |w|^{1+a_i}/(1+a_i), \tag{2.1}$$

near P_i. This is a special case of a cone-like singularity of a manifold where the metric takes the more general form

$$\sigma = dr^2 + r^2\sigma_N(r), \qquad 0 \le r < \varepsilon, \tag{2.2}$$

with a smooth family of nonsingular metrics $\sigma_N(r)$ on a smooth compact manifold N^m without boundary, referred to as the cross section of the cone, and $\sigma_N(r)$ constant for small r. In this special case, N is the circle $S^1_{1+a_i}$ of radius $1+a_i$. The spectral geometry of Riemannian spaces Y with cone-like singularities as such has been studied by Cheeger [C], Nagase [N], and Brüning-Seeley [BS]. Melrose [M] has also studied the spectral geometry of *proper conic* metrics defined by the existence of a defining function r, on a compact manifold with boundary, in terms of which

$$\sigma = r^{2s}((dr/r)^2 + h), \tag{2.3}$$

where h is a C^∞-symmetric 2-tensor. We note that by setting $w = re^{i\theta}$, (1.7) becomes a special case of (2.3) with $s = 1 + a_i$ and $h = d\theta^2$.

Let $X = \Sigma\backslash\{P_1, \dots, P_n\}$ and $H(X)$ denote the completion of

$$\left\{u \in C_0^\infty(X) : \|u\|_1^2 = \int_X (|\nabla u|^2 + |u|^2)\, dV < \infty\right\}, \tag{2.4}$$

in the metric induced by $\|u\|_1^2 = \|u\|^2 + \|\nabla u\|^2$. On $H(X)$ we consider the symmetric bilinear form given by

$$q(u, v) = \int_X (\nabla u . \nabla v)\, dV, \tag{2.5}$$

for $u, v \in H(X)$. Then q is a semibounded closed quadratic form. This gives q as the quadratic form of a unique selfadjoint operator $-\Delta$, i.e.,

$$(-\Delta u, v) = q(u, v), \tag{2.6}$$

where $u, v \in H(X)$.

In the general case, with Y described as above, one has

Theorem 2.1 [C, N]. *If $m = \dim N$ is odd, then the collection of eigenfunctions ϕ such that ϕ, $d\phi$, and $\Delta\phi \in L^2$ determines a complete orthonormal basis of L^2. The eigenspaces are finite dimensional and the eigenvalues satisfy $0 \le \lambda_1 \le \lambda_2 \cdots \to \infty$. Furthermore,*

$$N(\lambda) \sim \frac{\mathrm{vol}(Y)\lambda^{(m+1)/2}}{(4\pi)^{(m+1)/2}\Gamma((m+1)/2+1)}, \quad \text{as } \lambda \uparrow \infty,$$

where $N(\lambda) = |\{j \ge 1 : \lambda_j \le \lambda\}|$.

The proof of this theorem (and the corresponding statement if $\dim N$ is even) uses as a main technique the method of separation of variables and the solutions of Bessel equations with a regular singularity at $r = 0$. In the special case of the conic Laplacian on S^2, the discreteness of $\mathrm{spec}(\Delta)$ is pretty straightforward:

Lemma 2.2. *Let Δ be the Laplacian associated with a conical metric σ on S^2. Then* $\mathrm{spec}(\Delta)$ *is discrete.*

Proof. Let σ be a conical metric on S^2 with vertices $\{P_1, \dots, P_n\}$ and exponents $\{a_1, \dots, a_n\}$. Let B_i be the ball centered at P_i with radius r_i. Without loss of generality we assume $r_i = 1$. Let $X = S^2 \backslash \{P_1, \dots, P_n\}$. To show $\mathrm{spec}(\Delta)$ is discrete, we need only show

$$\mathbf{F}_X = \left\{ u \in C^\infty(X) : \int_X (|\nabla u|^2 + u^2)\, dV \le 1 \right\} \tag{2.7}$$

is precompact in $L^2(X)$: see [RS].

We show first that for any real number δ, the set

$$\mathbf{F}_{B_i} = \left\{ u \in C^\infty(B_i) : \int_{B_i} (|\nabla u|^2 + u^2)\, dV \le \delta,\ u = 0 \text{ on } \partial B_i \right\}$$

is precompact in $L^2(B_i)$. For notational convenience, we drop the subscript i. On B, the Laplacian associated with σ, acting on functions, is given by

$$\Delta = r^{-2a-2}[r\partial_r + r^2\partial_r^2 + \partial_\theta^2]. \tag{2.8}$$

Consider on B the eigenvalue problem

$$\begin{aligned} \Delta u + \lambda u &= 0, \\ u &= 0 \quad \text{on } \partial B. \end{aligned} \tag{2.9}$$

Set $u(r, \theta) = f(r)g(\theta)$ and $\alpha = a + 1$. Then (2.9) is equivalent to

$$\begin{aligned} r^2(f'' + f'/r + \lambda r^{2\alpha} f)/f &= -g''(\theta)/g(\theta), \\ f(1) &= 0. \end{aligned} \tag{2.10}$$

The first equation in (2.10) gives $g''(\theta)/g(\theta) = -m^2$, where m is a nonnegative integer. Thus $g(\theta) = \alpha \cos m\theta + \beta \sin m\theta$ for some α, β. Set $y = f(r)$. Then

$$y'' + (1/r)y' + (\lambda r^{2\alpha} - m^2/r^2)y = 0. \tag{2.11}$$

The solution of equation (2.11) is given by

$$y = c_1 J_\nu(\sqrt{\lambda} r^{\alpha+1}/(\alpha+1)) + c_2 J_{-\nu}(\sqrt{\lambda} r^{\alpha+1}/(\alpha+1)), \tag{2.12}$$

where c_1 and c_2 are constants, $\nu = m/(\alpha+1)$, and J_ν is the Bessel function of order ν. Since $u \in \mathbf{F}_B$, we have $c_2 = 0$. Thus (2.12) reduces to

$$y = c_1 J_\nu(\sqrt{\lambda} r^{\alpha+1}/(\alpha+1)).$$

The condition $f(1) = 0$ implies $\sqrt{\lambda}/(\alpha+1)$ is a zero of the function $J_\nu(x)$, which has infinitely many real zeroes $k_{m,j}$ $(j = 1, 2, 3, \dots)$. We can then write the eigenfunctions in the form

$$J_\nu(k_{m,j} r^{\alpha+1})(\alpha \cos m\theta + \beta \sin m\theta),$$

where α, β are arbitrary. Furthermore any function in $C^2(B)$ which vanishes on ∂B, can be expressed in a convergent series of the form

$$\sum_{m,j} a_{m,j} J_\nu(k_{m,j} r^{\alpha+1}) \cos m(\theta - \theta_{m,j}). \tag{2.13}$$

In particular, if $\{f_k\} \in \mathbf{F}_B$, then having normalized the eigenfunctions, we get

$$\sum_{m,j} (\lambda_{m,j}^2 + 1)|a_{m,j}^{(k)}|^2 \le \delta.$$

Thus for a fixed m, there is a sequence $\{k_l\}$ such that $\{a_{r,j}^{(k_l)}\}$ converges for all $r \le m$. Using the diagonal process, there exists $\{s_l\}$ such that $\{a_{m,j}^{(s_l)}\} \to b_{m,j}$ as $s_l \to \infty$. Set

$$f(r, \theta) = \sum_{m,j} b_{m,j} J_\nu(k_{m,j} r^{\alpha+1}) \cos m(\theta - \theta_{m,j}).$$

Then $\{f_k^{(s_l)}\} \to f$ in $L^2(B)$ and $\mathbf{F}_B$ is precompact in $L^2(B)$.

Now we can show that $\mathbf{F}_X$ is precompact in $L^2(X)$. Let $\{\phi_i\}$ be a partition of unity subordinate to $B_1, \dots, B_n, X \backslash U_{i=1}^n V_i$ where V_i is a closed subset of B_i and such that $|\nabla \phi_i| \le k$ for all i. Set $A_0 = X \backslash U_{i=1}^n V_i$, $A_i = B_i$ for $1 \le i \le n$. Then for $u \in \mathbf{F}_X$, $u = \sum_{i=0}^n \phi_i u_i$ and for every $0 \le i \le n$ we have

$$\begin{aligned}
\int_{A_i} (|\nabla(\phi_i u)|^2 + (\phi_i u)^2)\, dV &= \int_{A_i} (|\nabla u.\phi_i + u.\nabla\phi_i|^2 + \phi_i^2 u_i^2)\, dV \\
&\le \int_{A_i} (\phi_i^2 |\nabla u|^2 + 2\phi_i.u.|\nabla u|\,|\nabla\phi_i| + u^2|\nabla\phi_i|^2 + \phi_i^2 u^2)\, dV \\
&= \int_{A_i} \phi_i^2 (|\nabla u|^2 + u^2)\, dV + \int_{A_i} (|\nabla\phi_i|^2 u^2 + 2\phi_i u |\nabla u|\,|\nabla\phi_i|)\, dV \\
&\le \int_{A_i} (|\nabla u|^2 + u^2)\, dV + \int_{A_i} (k^2 u^2 + k(u^2 + |\nabla u|^2))\, dV \\
&\le 1 + k^2 + k,
\end{aligned}$$

whenever $\int_X (|\nabla u|^2 + u^2)\, dV \le 1$. Thus given $\{g_n\} \in \mathbf{F}_X$, one has $\{\phi_i g_n\} \in \mathbf{F}_{A_i}$ for all $0 \le i \le n$. Using the first result and a diagonal process we can extract a convergent subsequence $\{\phi_i g_{n_k}\}$ in $L^2(X)$, hence the desired claim. □

3. Asymptotics of the heat kernel

The authors in [BS] give a scheme for computing the asymptotics of $\mathrm{TR}(e^{-Lt})$ as $t \to 0^+$ for certain singular operators L, a principal example of which is the Laplace operator for a manifold with a singularity, where the metric takes the form (2.2). The idea there is to study an appropriate power of the resolvent of L then pass to the heat kernel by a contour integral. The Hilbert space L^2 on which Δ acts is a direct sum of an *interior* part H_i, consisting of those functions vanishing identically within some distance of the singularity, and a *boundary* part H_b, consisting of the complementary space. Since the interior part is well understood, the heart of the problem is thus the construction of the boundary parametrix. Let $W = \{w \in \mathbf{C} : |\arg w| < \pi - \varepsilon\}$, $\varepsilon > 0$. The main result, in our case, is that for any smooth function ϕ supported sufficiently near the singular points, we have

$$\mathrm{TR}[\phi(\Delta + z^2)^{-2}] = \int_0^\infty \sigma(x, xz)\, dx, \tag{3.1}$$

where $\sigma(x, w)$ defined on $\mathbf{R} \times W$ is C^∞ in x, with derivatives analytic in w, and such that $\sigma(x, w) \sim \sum_j \sigma_j(x) w^{\alpha_j}$ gives the following expansion as $w \to \infty$ in W:

$$\begin{aligned}\int_0^\infty \sigma(x, xz)\, dx \sim \sum_{k \geq 0} z^{-k-1} \int_0^\infty \frac{w^k}{k!} \sigma^{(k)}(0, w)\, dw \\ + \sum \int_0^\infty (zx)^{\alpha_j} \sigma_j(x)\, dx \\ + \sum_{\alpha_j = -1}^{-\infty} z^{\alpha_j} \ln z \frac{\sigma_j^{(-\alpha_j - 1)}(0)}{(-\alpha_j - 1)!}.\end{aligned} \tag{3.2}$$

Here $\{\alpha_j\}$ is a sequence of complex numbers with $\Re(\alpha_j) \to -\infty$, and the functions $\sigma^{(k)}(x, w) = \partial_x^k \sigma(x, w)$ and $\sigma_j \in S(\mathbf{R})$ (Schwartz class) are determined by any interior parametrix valid away from the singular point and the divergent integrals are defined by analytic continuation; see [BS].

To give an idea how these asymptotic expansions arise, the Laplacian in the neighborhood of a vertex P_i is given by

$$\Delta = -\partial_r^2 - r^{-1}\partial_r + r^{-2}\Delta_N, \tag{3.3}$$

where Δ_N is the Laplacian on $N = S^1_{1+a_i}$. The change of variables $f \to \sqrt{r} f$ transforms Δ into

$$\Delta = -\partial_r^2 + r^{-2}(\Delta_N - 1/4). \tag{3.4}$$

Thus on H_b, Δ is given by

$$-\partial_r^2 + r^{-2} A, \qquad A = \Delta_N - \tfrac{1}{4}, \tag{3.5}$$

where the operator A is unbounded on $L^2(N)$, with $A \geq -\frac{1}{4}$. The singular operator, D_a, which is the Friedrich's extension of $D = -\partial_r^2 + r^{-2} a$, $a \geq -\frac{1}{4}$, has resolvent with kernel

$$k_\nu(x, y, z) = (xy)^{1/2} I_\nu(xz) K_\nu(yz), \qquad x \leq y, \tag{3.6}$$

where $\nu = \sqrt{a + \frac{1}{4}}$, and I_ν, K_ν are Bessel functions: see [Ca]. If ϕ has compact support, then $\phi(D_a + z^2)^{-1}$ has finite trace given by an integral as in (3.2) where

$$\sigma(x, w) = x\phi(x) I_\nu(w) K_\nu(w) \tag{3.7}$$

has an expansion in terms of $w^{-1}, w^{-2}, \ldots$ as $w \to \infty$ in any open sector $|\arg w| < \pi/2 - \varepsilon$. A similar expansion for the trace of $\phi(\Delta + z^2)^{-2}$, can be obtained by using the kernel of $(D_a + z^2)^{-2}$, $a \in \operatorname{spec}(\Delta)$, which is given on the diagonal by

$$k_\nu^2(x, x, z) = \left(-\frac{1}{2z}\frac{\partial}{\partial z}\right)(xI_\nu(xz)K_\nu(xz)). \tag{3.8}$$

Let γ be a cut-off function supported sufficiently close to the singular point $x = 0$, such that $\Delta = \Delta_b$ on $\operatorname{supp}\gamma$ and $\gamma(x) \equiv 1$ for small x. Then, as shown in [BS], $\mathrm{TR}(\gamma e^{-\Delta t})$ has an asymptotic expansion, as $t \to 0$, in $t^{(k-3)/2}$, $t^{(-4-a)/2}$, and $t^{(-4-a)/2}\log t$. Here $a \leq -1$ and $k \geq 3$ are integers. Going back to (1.2) and (1.6), we observe that as far as $\det' \Delta$ is concerned the terms of interest are the constant term, c_0, and the coefficient of $\log t$, c_1, in this asymptotic expansion. Problems arise when $c_1 \neq 0$ since then $Z(s)$ has a first order pole at $s = 0$:

$$\int_0^1 t^{s-1}\log t\, dt = \lim_{\varepsilon\to 0}\int_\varepsilon^1 t^{s-1}\log t\, dt = \lim_{\varepsilon\to 0}\left[\frac{-\varepsilon^s\log\varepsilon}{s} - \frac{1}{s^2}\right] = -\frac{1}{s^2}.$$

Set

$$\zeta(s) = \sum_{a\in\operatorname{spec}(A)} \nu(a)^{-s}, \tag{3.9}$$

where $\nu(a) = \sqrt{a + \frac{1}{4}}$. Let B_k denote the kth Bernoulli number, and $\operatorname{Res}_k f(z_0)$ denote the coefficient of $(z - z_0)^{-k}$ in the Laurent expansion of a meromorphic function f at z_0. Then c_0 and c_1 are given by

Theorem 3.1 [BS].

(a)

$$c_0 = -\frac{1}{2}\operatorname{Res}_0\zeta(-1) - \frac{1}{4}\sum_{k\geq 1}(-1)^k\frac{B_k}{k}\operatorname{Res}_1\zeta(2k-1) + \frac{\Gamma'(-\frac{1}{2})}{8\sqrt{\pi}}\operatorname{Res}_1\zeta(-1) + \int_0^\infty g_0(x)\,dx,$$

(b) $c_1 = \frac{1}{4}\operatorname{Res}_1\zeta(-1)$, *where* $g_0(x)$ *satisfies* $\lim_{x\to 0} x g_0(x) = -\frac{1}{2}\operatorname{Res}_1\zeta(-1)$.

For $N = S^1_{1+a}$ and $\sigma_N(r) = (1+a)^2 d\theta^2$, if $\beta = 1 + a$ then the eigenvalues of A are given by $\{n^2/\beta^2 - 1/4\}$, $n \in \mathbf{Z}$, and the zeta function (3.9) is given by

$$\zeta(s) = 2\sum_{n=1}^\infty\left(\frac{n}{\beta}\right)^{-s} = 2\beta^s\zeta_R(s). \tag{3.10}$$

Since

$$\zeta_R(-1) = -1/12, \quad \operatorname{Res}_1\zeta_R(1) = 1, \quad \text{and} \quad B_1 = -1/6,$$

Theorem 3.1 yields $c_1 = 0$, and

$$c_0 = \frac{1}{12\beta} - \frac{\beta}{12} + \int_0^\infty g_0(x)\,dx. \tag{3.11}$$

Thus we have the regularized interior term $\int_0^\infty g_0(x)\,dx$, plus singular terms in the zeta function (3.9). One can easily show that if σ is flat on $\Sigma_0\backslash\{P_1, \ldots, P_n\}$

then $\int_0^\infty g_0(x)\,dx = 0$. Since $c_1 = 0$, $Z(s)$ is regular at $s = 0$ and we can use (1.2) to define $\det'\Delta$.

4. Real analyticity

Let $\mathscr{C}_n$ denote the space of conical metrics on the sphere with n vertices. Let $\mathscr{C}_n^*$ denote the subspace consisting of those metrics with fixed exponents. Any point in $\mathscr{C}_n$ determines a smooth conformal structure on S^2. Since there is only one conformal structure on S^2 determined by the standard round metric σ_0, it follows that each point in $\mathscr{C}_n$ has a representative of the form

$$\sigma = e^{2\phi}\sigma_0, \tag{4.1}$$

where ϕ is smooth and harmonic with respect to σ_0 on $S^2\backslash\{P_1, \dots, P_n\}$, for some $P_1, \dots, P_n$. We identify S^2 with $\overline{\mathbf{C}} = \mathbf{C}\cup(\infty)$ with local coordinates z on $\mathbf{C}$ and $1/z$ near ∞ and with the standard conformal structure. A conical metric σ on $\overline{\mathbf{C}}\backslash\{\tau_1, \dots, \tau_n\}$, with $\tau_i \in \overline{\mathbf{C}}$, corresponding to (4.1) may be written as $\sigma = e^{2\phi}|dz|^2$ on $\mathbf{C}\backslash\{\tau_1, \dots, \tau_n\}$, and as $\sigma = e^{2\phi(1/z)}|dz|^2/|z|^4$ for z near 0, using the coordinate $1/z$ near ∞. We shall show

Theorem 4.1. $\det'\Delta\colon \mathscr{C}_n^* \to \mathbf{R}$ *is real analytic.*

Proof. Let us first prove the theorem for the case of four vertices. Let σ be a conical metric on $\overline{\mathbf{C}}$ with vertices $\{\tau, \tau_0, \tau_1, \tau_2\}$ and fixed exponents $\{a, a_0, a_1, a_2\}$. Assume that τ_i is fixed for $i = 0, 1, 2$ with $|\tau_i| > 2$ and $|\tau| < \varepsilon$ for a given $\varepsilon > 0$. Rather than work with Δ_σ, a fixed operator on a varying space, it is more convenient to work with a family of varying operators on a fixed underlying space. To that end, let f be a C^∞-function with $f = 1$ inside the circle of radius 1 about 0 and $f = 0$ outside the circle of radius 2 about 0. Let $\{g_t(z)\}$ be the flow generated by the vector field $X = f\tau$ with $0 \le t \le 1$. Then $\phi_\tau(z) = g_1(z) = z + f\tau$ is a diffeomorphism of $\overline{\mathbf{C}}$ fixing τ_i and mapping the disk of radius 1 about 0 conformally to the disk of radius 1 about τ with $\phi_\tau(0) = \tau$. Under pullback of metrics, we get a family of metrics $\{\sigma_\tau\}$ on $\overline{\mathbf{C}}$ with fixed vertices $\{0, \tau_0, \tau_1, \tau_2\}$ which is analytic in τ and fixed in the neighborhood of a vertex.

Near a vertex τ of a conical metric on surface Σ, we have Figure 1.

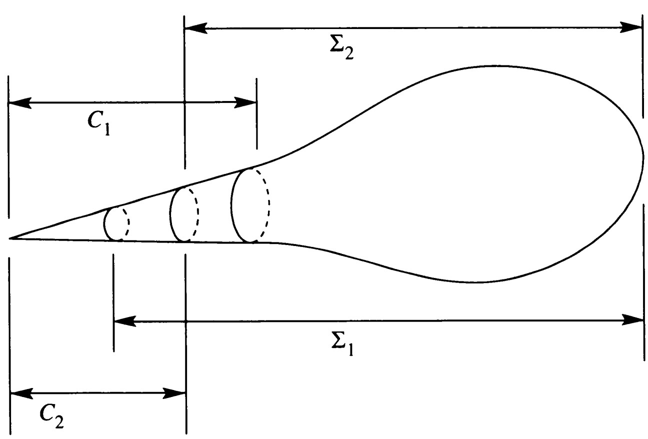

FIGURE 1

It is shown in [N, Lemma 5.3] that

$$k_{\tau,\Sigma}(x, x, t) = k_{\tau,\Sigma_1}(x, x, t)I_{\Sigma_2} + k_{\tau,C_1}(x, x, t)I_{C_2} + O(e^{-\delta/t}), \tag{4.2}$$

where $k_{\tau,M}$ is the heat kernel on M, I_M is the characteristic function on M and $O(e^{-\delta/t})$, $\delta > 0$ is a term any derivatives of which decrease exponentially when $t \to 0$. The main idea there is to reconstruct the heat kernel of Σ, using E. E. Levi's method, from the well-known heat kernel of a compact surface and from the formal representation of the heat kernel involving Bessel functions in a neighborhood of a cone-like singularity. Using estimates of Bessel functions and the uniqueness of the heat kernel, one gets (4.2).

Set $h(\sigma) = -\log(\det' \Delta)$, then

$$\begin{aligned} h(\sigma_\tau) = & \left.\frac{d}{ds}\right|_{s=0} \frac{1}{\Gamma(s)} \left[\int_0^1 \int_\Sigma k_{\tau,\Sigma}(x, x, t)\, dx\, t^s \frac{dt}{t}\right] \\ & + \int_1^\infty \int_\Sigma k_{\tau,\Sigma}(x, x, t)\, dx \frac{dt}{t}. \end{aligned} \tag{4.3}$$

To analyze (4.3) note that we have arranged the problem so that the metric does not change in the neighborhood of a vertex. Thus one parametrix for the Laplacian in a neighborhood of a vertex will work uniformly for all vertices, and the construction of such a parametrix has already been done in [BS]. Away from the vertices, we need a further result. Let $\{g_\tau\}$ be a family of metrics analytic in τ on a compact surface M with smooth boundary, and $\{\Delta_\tau\}$ be the corresponding family of Laplacians on $L^2(M)$. Let $\{\phi_{1,\tau}, \phi_{2,\tau}, \dots\}$ be a complete orthonormal basis of $L^2(M)$ consisting of eigenfunctions of Δ_τ with $\phi_{j,\tau}$ having eigenvalue $\lambda_{j,\tau} : 0 < \lambda_{1,\tau} \le \lambda_{2,\tau}, \dots$. Then $\lambda_{1,\tau}$ is continuous in τ and is uniformly bounded from below by some constant c. Let $k_\tau(x, x, t)$ denote the heat kernel on (M, g_τ). Then

Lemma 4.2. (1) *$k_\tau(x, x, t)$ is analytic in τ.*

(2) *Given $T > 0$, $|k_\tau(x, x, t)| = O(e^{-ct/4})$ for all $t \ge T$.*

(3) *For small t and fixed τ_0, the error term in the expansion of $k_\tau(x, x, t)$ is uniform in τ in a neighborhood of τ_0.*

Proof. We recall the construction of a parametrix for Δ. Let $\varepsilon = \operatorname{inj}(M)$, $B_\varepsilon = B(y, \varepsilon)$ for $y \in M$. We introduce geodesic spherical coordinates on B_ε by

$$x = \exp_y(r\xi), \qquad 0 \le r \le \varepsilon,\ \xi \in M_y \text{ with } |\xi| = 1.$$

Let

$$S_k(x, y, t) = \frac{1}{4\pi t} \exp\left(\frac{-d^2(x, y)}{4t}\right) \sum_{j=0}^{k} u_j(x, y) t^j, \tag{4.4}$$

where $u_j(\ , y) : B_\varepsilon \to \mathbf{R}$ satisfy

$$\begin{aligned} & u_0(y, y) = 1, \\ & \partial u_0/\partial r + \tfrac{1}{2}(\phi'/\phi)u_0 = 0, \\ & \partial u_j/\partial r + [\tfrac{1}{2}\phi'/\phi + j/r]u_j = \Delta u_j/r, \end{aligned} \tag{4.5}$$

for $j \geq 1$. Here we use the notation of [Ch, p. 149], where $r\phi(\exp_y r\xi, y)$ is defined as a determinant of a path of linear transformations on the orthogonal space of ξ defined in terms of parallel translation and Jacobi fields along geodesics γ_ξ. The solution to (4.5) is given by

$$u_0(x, y) = \phi^{-1/2}(x, y),$$
$$u_j(x, y) = u_0(x, y) \int_0^1 \tau^{j-1}(u_0 \Delta u_{j-1})(\exp_y \tau r\xi, y)\, d\tau, \tag{4.6}$$

for $j \geq 1$. Let $0 \leq \rho \leq 1 \in C^\infty(M \times M)$ be equal to 1 on $B_{\varepsilon/4}$ and 0 on $(M \times M) \backslash B_{\varepsilon/2}$. A parametrix for $(\Delta - \partial/\partial t)$ on M is then given by $H_k = \rho S_k$, which satisfies

$$(\Delta_x - \partial/\partial t) H_k = t^{k-1} \exp(-d^2(x, y)/4t) G_k, \tag{4.7}$$

with $G_k \in C^\infty(M \times M \times [0, \infty))$. A fundamental solution to the heat equation is then given by

$$k(x, x, t) = H_k + H_k * \sum_{l=1}^{\infty} \left[\left(\Delta_x - \frac{\partial}{\partial t} \right) H_k \right]^{*l} = H_k + H_k * F,$$

where for $0 \leq t \leq T$, one has the estimate

$$|F(x, y, t)| \leq C t^{k-1} \exp(-d^2(x, y)/4t). \tag{4.8}$$

It is well known from the theory of ordinary differential equations [CL] that the solution of an ordinary differential equation, depending continuously (or analytically) on a parameter, depends continuously (or analytically) on the given data. Since geodesics $\gamma : (a, b) \to M$ are solutions of second degree ordinary differential equations whose coefficients are the Christoffel symbols (which are smooth functions in the coefficients of the metric g_τ), geodesics on M depend smoothly on τ. Similarly for a parallel field of vectors $v(t)$ along a geodesic or for a Jacobi field Z. We can thus conclude that $u_{0,\tau}$ is analytic in τ. Using the iteration formula (4.6) we see that $u_{j,\tau}$ is analytic in τ for all j. This gives (1) in Lemma 4.2.

For the large time behavior, let $T > 0$ be given. Then

$$k_\tau(x, x, t) = \sum_{j=1}^{\infty} e^{-\lambda_{j,\tau} t} \phi_{j,\tau}(x) \phi_{j,\tau}(x), \tag{4.9}$$

and

$$\left| \sum_{j=1}^{\infty} e^{-\lambda_{j,\tau} t} \phi_{j,\tau}(x) \phi_{j,\tau}(x) \right| \leq \sum_{j=1}^{\infty} e^{-\lambda_{j,\tau} t} \|\phi_{j,\tau}\|_\infty^2$$
$$\leq C \sum_{j=1}^{\infty} e^{-\lambda_{j,\tau} t} \|D^2 \phi_{j,\tau}\|_2^2 \leq C \sum_{j=1}^{\infty} e^{-\lambda_{j,\tau} t} (\lambda_{j,\tau})^2,$$

where we used a standard Sobolev inequality [F]. Using the simple fact that $e^{-s} s \leq e^{-s/2}$ for $s \geq \ln 4$, and the well-known asymptotic distribution of large eigenvalues $\lambda_j \sim \alpha j$ as $j \to \infty$, we can choose N large enough so that for $j \geq N$ we have

$$e^{-\lambda_{j,\tau} t} (\lambda_{j,\tau})^2 \leq (2/t^2) e^{-\lambda_{j,\tau} t/4}, \tag{4.10}$$

$$\lambda_{j,\tau} \geq \alpha j/2, \tag{4.11}$$

and

(4.12) $$e^{-(\alpha j/2-\lambda_{1,\tau})t/4} \le j^{-2}.$$

Then

(4.13) $$\sum_{j=1}^{N} e^{-\lambda_{j,\tau}t}(\lambda_{j,\tau})^2 \le 2Ne^{-\lambda_{1,\tau}t/4}$$

and

(4.14) $$\begin{aligned}\sum_{j=N+1}^{\infty} e^{-\lambda_{j,\tau}t}(\lambda_{j,\tau})^2 &\le 2\sum_{j=N+1}^{\infty} e^{-\lambda_{j,\tau}t/4} \le 2e^{-\lambda_{1,\tau}t/4}\sum_{j=N+1}^{\infty} e^{-\mu_{j,\tau}t/4}\\ &\le 2e^{-\lambda_{1,\tau}t/4}\sum_{j=N+1}^{\infty} e^{-(\alpha j/2-\lambda_{1,\tau})t/4}\\ &\le 2e^{-\lambda_{1,\tau}t/4}\sum_{j=N+1}^{\infty} j^{-2} \le Be^{-\lambda_{1,\tau}t/4},\end{aligned}$$

where

$$\mu_{j,\tau} = \lambda_{j,\tau} - \lambda_{1,\tau}, \qquad B = 2\sum_{j=N+1}^{\infty} j^{-2}.$$

Putting all this together we get

$$|k_\tau(x, x, t)| \le C(2N+B)e^{-\lambda_{1,\tau}t/4},$$

hence the desired estimate

$$|k_\tau(x, x, t)| = O(e^{-ct/4}).$$

For the small time behavior, let $T > 0$ and τ_0 be given. Then, for $t \in [0, T]$, the constant C appearing in (4.8) is a function of the volume of M, T, and $\sup|G_k|$ on $M\times M\times[0, T]$. Using the analyticity of $u_{j,\tau}$ in τ on the compact surface M we get the uniformity in τ of $\sup|G_k|$ in a neighborhood of τ_0, hence the desired result. Putting together (4.2), (4.3), Lemma 4.2, and the remark preceding it we get Theorem 4.1 for four vertices.

For the general case, we proceed in a similar manner. Let σ be a conical metric on $\overline{\mathbf{C}}$ with fixed exponents and with vertices $\{0, 1, \infty, \tau_4, \dots, \tau_n\}$. Assume that $|\tau_i - \tau_i^0| < \varepsilon$ for a given ε and fixed τ_i^0. We construct, as before, a diffeomorphism of $\overline{\mathbf{C}}$ fixing $\{0, 1, \infty\}$ and mapping a small disk about τ_i conformally onto a small disk about τ_i^0, with τ_i mapped to τ_i^0. Under pullback of metrics, we get a family of metrics on $\overline{\mathbf{C}}$ with fixed vertices $\{0, 1, \infty, \tau_4^0, \dots, \tau_n^0\}$ which is analytic in the parameters τ_i and is fixed in the neighborhood of a vertex. The proof is now identical to the one for four vertices. This finishes the proof of the theorem. □

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